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Fermi Gas on a Lattice in the van Hove Limit

T. G. Ho[1] **and L. J. Landau**[2]

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We study a Fermi gas with general translation-invariant many-body interactions on a $(\nu \geqslant 3)$-dimensional lattice. A complete analysis is given of the perturbative terms up to second order and the program put forward by N. M. Hugenholtz for the derivative of the Boltzmann equation is verified to second order.

KEY WORDS: Fermi gas; van Hove limit; Boltzmann equation.

1. INTRODUCTION

A program to derive the Boltzmann equation for a Fermi gas in the van Hove limit within the framework of algebraic quantum theory for infinite systems was proposed by Hugenholtz.[1] His goal was to derive the following results:

1. The van Hove limit depends only on the two-point function of the initial state.
2. The limit state has vanishing n-point truncated correlation functions if $n > 2$. That is, the limit state is quasifree.
3. The two-point function of the limit state evolves, with respect to the rescaled time, according to a semigroup which satisfies a nonlinear quantum Boltzmann equation.
4. Boltzmann's H-theorem holds (increase of entropy density).

His approach was to control the large-time behavior of the individual terms in the Dyson perturbative expansion in the coupling strength g using properties of oscillatory integrals.

Difficulties with his approach are as follows (see ref. 2 and ref. 3, §11.2 and Appendix C):

[1] Department of Mathematics, University of Nottingham, Nottingham, England.
[2] Department of Mathematics, King's College, London, England.

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1. His method is perturbative, analyzing the series term-by-term without controlling the *sum*. The series would not be expected to have a uniform nonzero radius of convergence in the van Hove limit.
2. Even at the level of individual terms in the perturbative expansion there are difficulties controlling the multiple-time integrals. Hugenholtz based his analysis on the following erroneous claim:

Claim. Let

$$f(s)=\int_{-\pi}^{\pi} du_1 \cdots du_n \; g(u_1,..., u_n) \exp[iE(u_1,..., u_n)\, s]$$

with g twice continuously differentiable and periodic with period 2π in each of the variables, and E a nonconstant, infinitely differentiable periodic function. Then $s^2 f(s)$ vanishes at infinity.

In fact the large-s behavior of such an integral depends on the detailed structure of the function $E(u_1,..., u_n)$, in particular on the nature of the stationary point.[4]

Rather than using momentum-space methods as in ref. 1, we use l^p-decay properties of the one-particle lattice time evolution kernel $K_t(x)$. We study here the perturbative terms for a general interaction, and verify the Hugenholtz program in second-order perturbation theory.[3]

In our approach certain terms are required to be zero by explicit cancellation. In higher orders such cancellations will again be required, but we do not have a general mechanism to produce such cancellations.

1.1. The Model

Throughout we shall set Planck's constant $\hbar = 1$ and the mass $m = 1$. The number of space dimensions is $\nu \geqslant 3$.

The Observables. To each point $x \in \mathbb{Z}^\nu$ is associated a fermion creation operator $a(x)^*$ and destruction operator $a(x)$ satisfying the canonical anticommutation relations

$$\{a(x), a(y)^*\} = \delta_{x,y}, \qquad \{a(x), a(y)\} = 0 \tag{1}$$

[3] For the considerably simpler case of a noninteracting Fermi gas in the presence of random static impurities described by a random external potential, Hugenholtz's program can be carried out and the sum of the series controlled for a Gaussian random potential and small rescaled time τ.[3]

We will denote the Fourier-transformed operators by

$$\hat{a}(k)^* = (2\pi)^{-\nu/2} \sum_{x \in \mathbb{Z}^\nu} a(x) \cdot e^{ik \cdot x}$$
$$\hat{a}(k) = (2\pi)^{-\nu/2} \sum_{x \in \mathbb{Z}^\nu} a(x)\, e^{-ik \cdot x} \tag{2}$$

where $k \in [-\pi, \pi]^\nu$. The Fourier-transformed operators satisfy

$$\{\hat{a}(q), \hat{a}(p)^*\} = \delta(p-q) \tag{3}$$

An observable is a localized expression in creation and annihilation operators of the form

$$A = \sum_{y_1, ..., y_{m_0} \in \mathbb{Z}^\nu} \Phi_0(y_1, ..., y_{m_0})\, a(y_1)^\# \cdots a(y_{m_0})^\# \tag{4}$$

with

$$\|\Phi_0\|_1 = \sum_{y_1, ..., y_{m_0} \in \mathbb{Z}^\nu} |\Phi_0(y_1, ..., y_{m_0})| < \infty$$

The order of A is $|A| = m_0$. Here $a^\#$ denotes a creation or annihilation operator. We shall only consider gauge-invariant observables, so A contains the same number of creation and annihilation operators.

The Hamiltonian. The Hamiltonian is

$$H = H_0 + gV \tag{5}$$

where

$$H_0 = \sum_{x, y \in \mathbb{Z}^\nu} h(x, y)\, a(x)^*\, a(y) = \int_{-\pi}^{\pi} d^\nu p\, \varepsilon(p)\, a(p)^*\, a(p) \tag{6}$$

Here

$$h(x, y) = -1/2\Delta_L(x, y) = \begin{cases} \nu & \text{if } \; x = y \\ -1/2 & \text{if } \; |x-y| = 1 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

and the energy $\varepsilon(k)$ is a sum of the one-dimensional energies,

$$\varepsilon(k) = (1 - \cos k_1) + \cdots + (1 - \cos k_\nu) \tag{8}$$

for $k = (k_1, k_2, ..., k_\nu)$.

The interaction strength is g and the translation-invariant self-interaction V has the form

$$V = \sum_{x \in \mathbb{Z}^\nu} W(x) \tag{9}$$

where $W(x)$ is the translate by x of the gauge-invariant interaction density W,

$$W = \sum_{y_1,\dots,y_m \in \mathbb{Z}^\nu} \Phi(y_1,\dots, y_m)\, a(y_1)^\# \cdots a(y_m)^\# \tag{10}$$

with $\|\Phi\|_1 < \infty$. Hence

$$V = \sum_{x \in \mathbb{Z}^\nu} \sum_{y_1,\dots,y_m \in \mathbb{Z}^\nu} \Phi(y_1,\dots, y_m)\, a(y_1 + x)^\# \cdots a(y_m + x)^\# \tag{11}$$

The order of V is $|V| = |W| = m$.

An important special case is the two-body interaction,

$$V = 1/2 \sum_{x,y \in \mathbb{Z}^\nu} \Phi(y-x)\, a(x)^*\, a(y)^*\, a(y)\, a(x) \tag{12}$$

where $\Phi(x) = \Phi(-x)$. The free time evolution is given by $A_t = \alpha_t^0(A)$, where

$$\begin{aligned} a_t(x) &= \alpha_t^0 a(x) = e^{itH_0} a(x)\, e^{-itH_0} \\ &= \sum_{y \in \mathbb{Z}^\nu} K_t(y-x)\, a(y) \end{aligned} \tag{13}$$

and

$$a_t(x)^* = \sum_{y \in \mathbb{Z}^\nu} \overline{K_t(y-x)}\, a(y)^* \tag{14}$$

We write Eqs. (13) and (14) more compactly as

$$a_t(x)^\# = \sum_{y \in \mathbb{Z}^\nu} K_t(y-x)^\#\, a(y)^\# \tag{15}$$

where $K_t^\#$ stands for either K_t or $\bar{K}_t$.

The full time evolution is $\alpha_t^g(A)$ and is given by the Dyson perturbative expansion

$$\alpha_t^g(A) = \sum_{n=0}^{\infty} (ig)^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n\, [V_{t_n}, [\dots, [V_{t_1}, A_t],\dots]] \tag{16}$$

The State. It is supposed that the initial state S of the Fermi gas is gauge-invariant, translation-invariant, and l^1-clustering. The clustering property is formulated in terms of the truncated correlation functions:

$$\gamma_p(x_1,..., x_p) = S^T(a(0)^\#, a(x_1)^\#,..., a(x_p^\#)) \tag{17}$$

which for an l^1-clustering state satisfy

$$\sum_{x_1,..., x_p} |\gamma_p(x_1,..., x_p)| < \infty \tag{18}$$

In the case of the two-point function we define

$$\hat{\gamma}(k) = \sum_x \gamma_1(x)\, e^{ik\cdot x} \tag{19}$$

Then

$$S(\hat{a}(k)^*\, \hat{a}(l)) = \gamma(k)\, \delta(k-l) \tag{20}$$

1.2. Goals and Results

The van Hove limit is defined by rescaling the coupling strength g to λg, introducing a rescaled time $\tau = \lambda^2 t$, and taking the limit $\lambda \to 0$. The expectation value of the observable A in the state S evolved to rescaled time τ with rescaled coupling strength λg is $S(\alpha^{\lambda g}_{\lambda^{-2}\tau} A)$.

According to the Hugenholtz program, one would like to show that as $\lambda \to 0$:

(a) The time-evolved state converges to an asymptotic state:

$$S(\alpha^{\lambda g}_{\lambda^{-2}\tau} A) \to \mathfrak{S}^g_\tau(A)$$

where the asymptotic state $\mathfrak{S}^g_\tau$ depends only on the two-point function of the initial state S.

(b) The asymptotic state $\mathfrak{S}^g_\tau$ is quasifree,

$$\mathfrak{S}^g_\tau(a(y_r)^* \cdots a(y_1)^*\, a(z_1) \cdots a(z_r))$$
$$= \sum_\Pi \sigma(\Pi)\, \mathfrak{S}^g_\tau(a(y_1)^*\, a(z_{\Pi(1)}))_t \cdots \mathfrak{S}^g_\tau(a(y_r)^*\, a(z_{\Pi(r)}))$$

where the summation is over all permutations Π of $\{1, 2,..., r\}$ and $\sigma(\Pi)$ is the sign of the permutation Π.

(c) The asymptotic two-point function γ_τ^g satisfies the irreversible nonlinear quantum Boltzmann equation (43) in the case of the two-body interaction (12).

We have developed general methods which are useful in studying (a)–(c) in perturbation theory. We have verified that (a)–(c) hold to second order in the coupling strength g.

The results of the paper can be summarized by the following theorems.

Theorem 1. Let S be a translation-invariant, l^1-clustering, gauge-invariant state. Let $\hat{S}$ be the gauge-invariant, quasifree state with the same two-point function as S. To zeroth order, the time evolution is just the free evolution given by $S_t = S \circ \alpha_t^0$. The time dependence of the state S under the free evolution satisfies the following:

1. If $n \geqslant 4$, for any $\delta > 0$, there is a constant C_δ such that

$$|S_t^T(a(x_1)^\#, \dots, a(x_n)^\#)| \leqslant C_\delta (1 + |t|)^{-n\nu/4 + \delta}$$

2. If $A = a(x_1)^\# \cdots a(x_n)^\#$, then $(|S_t(A) - \hat{S}(A)| \leqslant C_\delta (1 + |t|)^{-\nu + \delta}$.
3. $\lim_{t \to \infty} S_t = \hat{S}$ in the weak*-topology on $\mathfrak{A}^*$.

Theorem 2. Let S be a gauge-invariant, translation-invariant, and l^1-clustering state, and both A and V are gauge-invariant. Then the first-order perturbation terms of (22) are zero in the van Hove limit.

Theorem 3. The van Hove limit of the second-order term in the Dyson expansion exists and satisfies the quasifree condition (b) to second order in the coupling constant g. All contributions to the second-order terms from higher order truncated functions of the initial state S tend to zero.

Theorem 4. The Boltzmann equation [cf. (c)] is satisfied to second order in the coupling constant g in the case of the two-body interaction (12).

2. UNIFORM BOUNDS FOR THE LATTICE KERNEL

In one dimension the free lattice evolution K_t is given by

$$K_t^{(1)}(x) = (2\pi)^{-1} \int_{-\pi}^{\pi} dk\, e^{-i\varepsilon(k)t} e^{ixk} = e^{it} i^x J_x(t) \tag{21}$$

where $\varepsilon(k) = 1 - \cos k$, and $J_n(t)$ is the nth-order Bessel function of the first kind.

In ν dimensions, $K_t(x) = K_t^{(1)}(x_1) \cdots K_t^{(1)}(x_\nu)$ for $x = (x_1, ..., x_\nu)$.

Essential to our discussion are l^p-bounds on the Bessel functions. In ref. 5 uniform bounds on the Bessel functions are given as[4]

$$|J_n(t)| \leqslant 0.7858\ |t|^{-1/3}$$

$$\sum_{n=-\infty}^{\infty} |J_n(t)|^4 \leqslant \frac{2}{\pi} \frac{\log(1+\pi t)}{t}$$

Using these estimates, we can prove the following proposition (see also ref. 3).

Proposition 5. 1. $\|K_t\|_2 = 1$.

2. $\|K_t\|_\infty \leqslant A(1+|t|)^{-\nu/3}$ for some constant $A > 0$.
3. For any $\delta > 0$ there is a constant B_δ such that $\|K_t\|_4 \leqslant B_\delta(1+|t|)^{-\nu/4+\delta}$.
4. For any $\delta > 0$ there is a constant C_δ such that $\|K_t\|_3 \leqslant C_\delta(1+|t|)^{-\nu/6+\delta}$.
5. For any $\delta > 0$ there is a constant E_δ such that $\|K_t\|_p \leqslant E_\delta(1+|t|)^{-\nu/4+\delta}$ for all $p \geqslant 4$.

3. THE PERTURBATIVE TERMS

We are considering the time evolution of the state $S_t^{\lambda g}$ given by

$$S_t^{\lambda g}(A) = S(\alpha_t^\lambda(A))$$

$$= S\left(\sum_{n=0}^{\infty} (i\lambda g)^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n\, [V_{t_n}, [..., [V_{t_1}, A_1],...]]\right) \qquad (22)$$

We will assume the initial state S to be gauge-invariant, translation-invariant, and l^1-clustering, V of the form (11), and A of the form (4). In order to simplify the presentation, we will supress inessential indices. In (11) we supress the indices on the y-variables since the sum over $y_1, y_2, ..., y_m$ is controlled by $\Phi(y_1, y_2, ..., y_m)$ and will be estimated by $\|\Phi\|_1$ times an upper bound of the remaining factors which is independent

[4] The asymptotic behavior for large t is in fact[6]

$$\sum_{n=-\infty}^{\infty} |J_n(t)|^4 \sim \frac{1}{\pi^2} \frac{\log t}{t}$$

of the y-variables. Letting y represent any and all the y-variables, we can abbreviate (11) as

$$V=\sum_{y} \Phi(y) \sum_{x} \underbrace{a(x+y)^{\#} \cdots a(x+y)^{\#}}_{m \text{ factors}} \tag{23}$$

where each occurrence of x is shifted by a different y-variable. Similarly (4) becomes (setting $x_0=0$)

$$\sum_{y} \Phi_0(y) \underbrace{a(x_0+y)^{\#} \cdots a(x_0+y)^{\#}}_{m_0 \text{ factors}} \tag{24}$$

where each occurrence of x_0 is shifted by a different y-variable.

After substituting (23) and (24) in (22) we expand the commutators giving 2^n terms. The state S is then expanded as a sum of products of truncated functions S^T. A typical truncated function is of the form ($t_0=t$)

$$\begin{aligned} &S^T(a_{t_{j_1}}(x_{j_1}+y)^{\#},..., a_{t_{j_r}}(x_{j_r}+y)^{\#}) \\ &\quad= \sum_{z_1,..., z_r \in \mathbb{Z}^v} S^T(a(z_1)^{\#},..., a(z_r)^{\#})\, K_{t_{j_1}}(z_1-x_{j_1}-y)^{\#} \cdots K_{t_{j_r}}(z_r-x_{j_r}-y)^{\#} \\ &\quad= \sum_{u, w_1,..., w_{r-1}} \gamma_{r-1}(w_1,..., w_{r-1})\, K_{t_{j_1}}(u-x_{j_1}-y)^{\#} \\ &\qquad \times K_{t_{j_2}}(u-x_{j_2}-y+w_1)^{\#} \cdots K_{t_{j_r}}(u-x_{j_r}-y+w_{r-1})^{\#} \end{aligned} \tag{25}$$

where $u=z_1$, $w_j=z_{j+1}-z_1$, and $j_k \in \{0, 1,..., n\}$ for $k=1, 2,..., r$.

Again we simplify the expression by supressing inessential indices, as the sum over w-variables is controlled by $\gamma_{r-1}(w_1,..., w_r)$ and will be estimated by $\|\gamma_{r-1}\|_1$ times an upper bound of the remaining factors which is independent of the w-variables. Let w represent any and all the w-variables or zero. Then (25) can be abbreviated as

$$\begin{aligned} &S^T(a_{t_{j_1}}(x_{j_1}+y)^{\#},..., a_{t_{j_r}}(x_{j_r}+y)^{\#}) \\ &\quad=\sum_{w} \gamma_{r-1}(w) \sum_{u} K_{t_{j_1}}(u-x_{j_1}+w-y)^{\#} \cdots K_{t_{j_r}}(u-x_{j_r}+w-y)^{\#} \end{aligned} \tag{26}$$

where each x_j is shifted by a different w-variable (or zero) and a different y-variable. In the special case of the two-point function ($r=2$), the sum over u in (26) can be carried out giving

$$S^T(a_{t_{j_1}}(x_{j_1}+y)^*, a_{t_{j_2}}(x_{j_2}+y))$$
$$=\sum_w \gamma_1(w)\, K_{t_{j_2}-t_{j_1}}(x_{j_1}-x_{j_2}+w-y) \tag{27}$$

where y now also denotes a difference of y-variables.

Consider a typical nth-order term of the perturbative expansion consisting of:

1. α_{ij} two-point contributions with times t_i and t_j, where $i<j$, $i, j=0, 1,..., n$.
2. μ truncated functions labeled 1 to μ, with the jth truncated function of order $\beta_{j0}+\cdots+\beta_{jn}\geqslant 4$ with β_{jk} creation–annihilation operators at time t_k, $k=0, 1, 2,..., n$.

An upper bound for such a term is given by the product of $\|\Phi_0\|_1\, \|\Phi\|_1^n\, \|\gamma_{r_1-1}\|_1\cdots\|\gamma_{r_\mu-1}\|_1$, where $r_1+\cdots+r_\mu=mn+m_0$, times the supremum over y's and w's of

$$(\lambda g)^n \int_0^{|t|} dt_1\cdots\int_0^{t_{n-1}} dt_n \sum_{x_1,..., x_n}\sum_{u_1,..., u_\mu}\prod_{0\leqslant i<j\leqslant n} |K_{t_i-t_j}(x_j-x_i+w-y)|^{\alpha_{ij}}$$
$$\times \prod_{1\leqslant j\leqslant\mu,\, 0\leqslant k\leqslant n} |K_{t_k}(u_j-x_k+w-y)|^{\beta_{jk}}\, |\mathfrak{F}(t, t_1,..., t_n)| \tag{28}$$

where $\mathfrak{F}(t, t_1,..., t_n)$ is the product of all the truncated functions, each of which contains only operators all at the same time.

This expression (28) will be bounded by l^1-norms of the K-functions, and these norms are translation-invariant and hence do not depend on the w- and y-variables appearing in the argument of the K-function. We will supress the w- and y-variables, obtaining the final abbreviated expression for the upper bound:

$$\leqslant (\lambda g)^n\, C\int_0^{|t|} dt_1\cdots\int_0^{t_{n-1}} dt_n \sum_{x_1,..., x_n}\sum_{u_1,..., u_\mu}\prod_{0\leqslant i<j\leqslant n} |K_{t_i-t_j}(x_j-x_i)|^{\alpha_{ij}}$$
$$\times \prod_{1\leqslant j\leqslant\mu,\, 0\leqslant k\leqslant n} |K_{t_k}(u_j-x_k)|^{\beta_{jk}}\, |\mathfrak{F}(t, t_1,..., t_n)| \tag{29}$$

Fig. 1. Creation–annihilation oeprators.

We will bound $|\mathfrak{F}|$ by a constant as $S^T(a_{t_j}(x_1)^\# \cdots (a_{t_j}(x_r)^\#)$ is bounded independent of t_j and $x_1,\ldots, x_r$ since $\|a_t(x)^\#\| = 1$. We will use a better bound when we discuss the contributory terms.

4. DIAGRAMMATIC REPRESENTATIONS

A general N-point truncated function is represented by a bubble with directed lines joining the various points $t, t_1,\ldots, t_n$. Each line represents a creation–annihilation operator, a creation operator at time t_i by a line into t_i and an annihilation operator at time t_i by a line out of t_i (see Fig. 1). In the case of the two-point function the bubble is not drawn (see Figs. 2 and 3).

5. ZEROTH-ORDER TERMS

The zeroth-order term in the Dyson expansion (22) corresponds to the free evolution. We will now prove Theorem 1:

Proof of Theorem 1. 1. By (25), for $n \geqslant 4$,

$$\begin{aligned}
&|S_t^T(a(x_1)^\#,\ldots, a(x_n)^\#)| \\
&\quad = \left| \sum_{u, w_1,\ldots, w_{n-1}} \gamma_{n-1}(w_1,\ldots, w_{n-1})\, K_t(u-x_1)^\# \cdots K_t(u-x_n+w_{n-1})^\# \right| \\
&\quad \leqslant \|\gamma_{n-1}\|_1\, \|K_t\|_n^n \\
&\quad \leqslant C_\delta(1+|t|)^{-n\nu/4+\delta}
\end{aligned}$$

by Proposition 5.4.

2. Since $S_t(a(x)^*\, a(y)) = S(a(x)^*\, a(y))$, the result follows from 1.
3. This result follows from 2. ∎

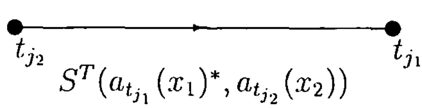

Fig. 2. Two-point function.

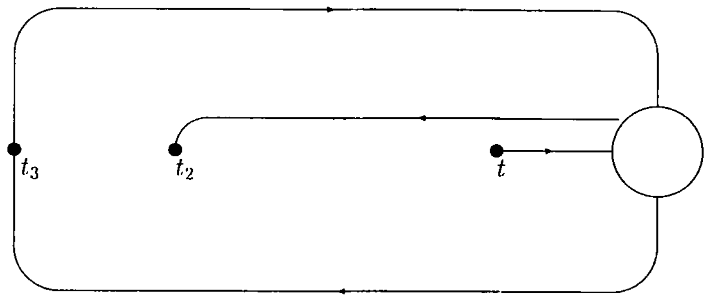

Fig. 3. $S^T(a_{t_1}(x_1)^*, a_{t_2}(x_2)^*, a_{t_3}(x_3), a_t(x_4))$.

6. FIRST- AND SECOND-ORDER TERMS IN THE LATTICE MODEL

In this section we show that in second-order perturbation theory in the van Hove limit (1) the limiting state depends only on the two-point function of the initial state, and (2) the limiting state has vanishing N-point truncated functions for $N > 2$.

By using Proposition 5, in particular,

$$\|K_t\|_3 \leqslant C_\delta(1+t)^{-\nu/6+\delta}$$
$$\|K_t\|_p \leqslant C_\delta(1+t)^{-\nu/4+\delta} \qquad \text{for} \quad p \geqslant 4$$

we will give estimates for the first- and second-order terms for $\nu \geqslant 3$ and $t > 0$. To ease future calculations we use the following lemmas:

Lemma 6. Let S be a translation-invariant state. Then

$$S\left(\left[\sum_{x \in \mathbb{Z}^\nu} W(x), A\right]\right) = 0$$

if $A = a(u)^* a(v)$ and $W = a(y)^* a(z)$.

Proof. We have

$$\begin{aligned} S&\left(\left[\sum_{x \in \mathbb{Z}^\nu} W(x), A\right]\right) \\ &= S\left(\left[\sum_{x \in \mathbb{Z}^\nu} a(y+x)^* a(z+x), a(u)^* a(v)\right]\right) \\ &= S(a(u+y-z)^* a(v) - a(u)^* a(v-y+z)) \\ &= \gamma_1(v-u-y+z) - \gamma_1(v-y+z-u) = 0 \quad \blacksquare \end{aligned}$$

Remark 7. If S is just a translation-invariant linear functional, the same argument applies. In particular, we conclude: Let S be a translation-invariant state. Then

$$S\left(\left[\sum_{x'\in\mathbb{Z}^\nu} W'(x'), \left[\sum_{x\in\mathbb{Z}^\nu} W(x), A\right]\right]\right)=0$$

if $A=a(u)^* a(v)$, $W=a(y)^* a(z)$, and $W'=a(y_1)^\# \cdots a(y_m)^\#$.

Lemma 8. Let $\hat{S}$ be a gauge-invariant and translation-invariant quasifree state. Then

$$\sum_{x\in\mathbb{Z}^\nu} \hat{S}([W(x), A])=0$$

if $A=a(u)^* a(v)$ and $W=a(y_1)^* \cdots a(y_m)^* a(z_1)\cdots a(z_m)$.

Proof. We have

$$\begin{aligned}
&\hat{S}\left(\sum_x [W(x), A]\right)\\
&\quad=\hat{S}\left(\sum_x [W, A(x)]\right)\\
&\quad=\hat{S}\left(\sum_x a(y_1)^* \cdots a(y_m)^* a(z_1)\cdots a(z_m)\, a(u+x)^* a(v+x)\right)\\
&\qquad-\hat{S}\left(\sum_x a(u+x)^* a(v+x)\, a(y_1)^* \cdots a(y_m)^* a(z_1)\cdots a(z_m)\right)\\
&\quad=\sum_{i,j=1}^{m} (-1)^{m+i+j+1}\, \hat{S}(a(y_1)^* \cdots a(y_{i-1})^* a(y_{i+1})^* \cdots a(y_m))\\
&\qquad\times\{\hat{S}(a(y_i)^* a(v+x))[\delta(z_j, u+x)-\hat{S}(a(u+x)^* a(z_j))]\\
&\qquad-\hat{S}(a(u+x)^* a(z_j))[\delta(v+x, y_i)-\hat{S}(a(y_i)^* a(y+x))]\}\\
&\quad=\sum_{i,j=1}^{m} (-1)^{m+i+j+1}\, \hat{S}(a(y_1)^* \cdots a(y_{i-1})^* a(y_{i+1})^* \cdots a(y_m))\\
&\qquad\times\left\{\gamma_1(v-u+z_1-y_1)-\sum_x \gamma_1(v+x-y_i)\,\gamma_1(z_j-u-x)\right.\\
&\qquad\left.-\gamma_1(v-u+z_1-y_1)+\sum_x \gamma_1(v+x-y_i)\,\gamma_1(z_j-u-x)\right\}=0
\end{aligned}$$

where we have used

$$\hat{S}(a(x)\, a(y)^*) = \delta(x, y) - \hat{S}(a(y)^*\, a(x)) \quad \blacksquare$$

Lemma 9. Let S be a translation-invariant state. Then

$$S\left(\left[\sum_{x \in \mathbb{Z}^\nu} W_t(x), A_s\right]\right) = S\left(\left[\sum_{x \in \mathbb{Z}^\nu} W_t(x), A_t\right]\right)$$

if $A = a(u)^*\, a(v)$ and $W = a(y_1)^\# \cdots a(y_m)^\#$.

Proof. Note that for any $B \in \mathfrak{A}$,

$$\begin{aligned}
&\left[B, \sum_{x \in \mathbb{Z}^\nu} a_s(u-x)^*\, a_s(v-x)\right] \\
&\quad = \left[B, \sum_{x, u_1, v_1} \overline{K_s(u_1 - u + x)}\, K_s(v_1 - v + x)\, a(u_1)^*\, a(v_1)\right] \\
&\quad = \left[B, \sum_{u_1} a(u_1)^*\, a(v + u_1 - u)\right] \\
&\quad = \left[B, \sum_{x \in \mathbb{Z}^\nu} a(u-x)^*\, a(v-x)\right]
\end{aligned}$$

which is independent of s. So

$$\left[B, \sum_{x \in \mathbb{Z}^\nu} a(u-x, s)^*\, a(v-x, s)\right] = \left[B, \sum_{x \in \mathbb{Z}^\nu} a(u-x, t)^*\, a(v-x, t)\right]$$

for any $s, t \in \mathbb{R}$. Thus

$$\begin{aligned}
&S\left(\left[\sum_{x \in \mathbb{Z}^\nu} W_t(x), a_s(u)^*\, a_s(v)\right]\right) \\
&\quad = S\left(\left[W_t, \sum_{x \in \mathbb{Z}^\nu} a_s(u-x)^*\, a_s(v-x)\right]\right) \\
&\quad = S\left(\left[W_t, \sum_{x \in \mathbb{Z}^\nu} a_t(u-x)^*\, a_t(v-x)\right]\right) \\
&\quad = S\left(\left[\sum_{x \in \mathbb{Z}^\nu} W_t(x), a_t(u)^*\, a_t(v)\right]\right) \quad \blacksquare
\end{aligned}$$

Remark 10. In the first-order case

$$S\left(\sum_x [W_{t_1}(x), A_t]\right)$$

we have from the structure of the commutator that there is at least one two-point contribution, that is, $\alpha_{01} \geqslant 1$. In the nth-order case

$$S\left(\sum_{x_1,\dots,x_n} [W_{t_n}(x_n),\dots, [W_{t_1}(x_1), A_t],\dots]\right)$$

we have

$$\sum_{i=0}^{j-1} \alpha_{ij} \geqslant 1 \qquad \text{for all} \quad 1 \leqslant j \leqslant n$$

6.1. First-Order Terms

In this subsection we will prove Theorem 2.

Proof of Theorem 2. Consider the first-order perturbative terms of (22),

$$i\lambda g \int_0^t dt_1\, S([V_{t_1}, A_t])$$

From Eq. (29) a first-order term is bounded by

$$\begin{aligned}\leqslant \lambda g C \int_0^t dt_1 \sum_{x_1} \sum_{u_1,\dots,u_\mu} &|K_{t-t_1}(x_1)|^{\alpha_{01}} \\ \times |K_t(u_1)|^{\beta_{10}} |K_{t_1}(u_1 - x_1)|^{\beta_{11}} \cdots &|K_t(u_\mu)|^{\beta_{\mu 0}} |K_{t_1}(u_\mu - x_1)|^{\beta_{\mu 1}} \end{aligned} \tag{30}$$

In (30) if we sum over the u_j first and then over x_1 we obtain

$$\lambda g C \int_0^t dt_1\, \|K_t\|_{r_1}^{\beta_{10}} \|K_{t_1}\|_{r_1}^{\beta_{11}} \cdots \|K_t\|_{r_\mu}^{\beta_{\mu 0}} \|K_{t_1}\|_{r_\mu}^{\beta_{\mu 1}} \|K_{t-t_1}\|_{\alpha_{01}}^{\alpha_{01}} \tag{31}$$

where $r_j = \beta_{j0} + \beta_{j1} \geqslant 4$ for $j = 1, 2,\dots, \mu$.

Note that for all $x \in \mathbb{Z}^\nu$,

$$\sum_u F(u)\, G(u - x) = \sum_u F(u + x)\, G(u) \tag{32}$$

If we label (30) using

$$\gamma_j = \min\{\beta_{j0}, \beta_{j1}\} \geqslant 1$$
$$\delta_j = \max\{\beta_{j0}, \beta_{j1}\} \geqslant 2$$
$$s_j = \begin{cases} t & \text{if } \gamma_j = \beta_{j0} \\ t_1 & \text{otherwise} \end{cases}$$
$$S_j = \begin{cases} t_1 & \text{if } \gamma_j = \beta_{j0} \\ t & \text{otherwise} \end{cases}$$

for $j = 1,..., \mu$ and shift the x_1 dependence to K_{s_j} by (32), then (30) becomes

$$\begin{aligned} \lambda g C \int_0^t dt_1 \sum_{x_1} \sum_{u_1,..., u_\mu} &|K_{t-t_1}(x_1)|^{\alpha_{01}} \\ &\times |K_{s_1}(u_1 - x_1)|^{\gamma_1} \cdots |K_{s_\mu}(u_\mu - x_1)|^{\gamma_\mu} |K_{S_1}(u_1)|^{\delta_1} \cdots |K_{S_\mu}(u_\mu)|^{\delta_\mu} \\ \leqslant \lambda g C \int_0^t dt_1 \, &\|K_{t-t_1}\|_r^{\alpha_{01}} \|K_{s_1}\|_r^{\gamma_1} \cdots \|K_{s_\mu}\|_r^{\gamma_\mu} \|K_{S_1}\|_{\delta_1}^{\delta_1} \|K_{S_\mu}\|_{\delta_\mu}^{\delta_\mu} \end{aligned} \tag{33}$$

with $r = \alpha_{01} + \gamma_1 + \cdots + \gamma_\mu$ and where we have summed over x_1 first and then the u_j.

By Remark 10, $\alpha_{01} \geqslant 1$, we therefore consider the following cases:

1. *If* $\alpha_{01} > 2$ *and* $\mu = 0$, then by (31), (30) is bounded by

$$\leqslant \lambda g C \int_0^t dt_1 \, \|K_{t-t_1}\|_{\alpha_{01}}^{\alpha_{01}}$$
$$\leqslant \lambda g C_\varepsilon \int_0^t dt_1 \, (1 + t - t_1)^{-\nu/2 + \varepsilon} \to 0$$

in the van Hove limit (We use ε to denote a positive constant which can be made arbitrarily small).

2. *If* $\alpha_{01} \geqslant 2$ *and* $\mu \geqslant 1$, then, as $\beta_{j0} + \beta_{j1} \geqslant 4$, $j = 1, 2,..., \mu$, by (31), (30) is bounded by

$$\leqslant \lambda g C \int_0^t dt_1 \, \|K_{t-t_1}\|_{\alpha_{01}}^{\alpha_{01}} \|K_t\|_{r_1}^{\beta_{10}} \|K_{t_1}\|_{r_1}^{\beta_{11}} \cdots \|K_t\|_{r_\mu}^{\beta_{\mu 0}} \|K_{t_1}\|_{r_\mu}^{\beta_{\mu 1}}$$
$$\leqslant \lambda g C_\varepsilon \int_0^t dt_1 \, (1 + t_1)^{-\nu/2 + \varepsilon} \to 0$$

in the van Hove limit.

3. *If* $\alpha_{01}=2$ *and* $\mu=0$, then such terms can be written as

$$\sum_{x} S([W'(x), A'])\,\mathfrak{F}(t, t_1)$$

where $|W'|=|A'|=2$ and $\mathfrak{F}$ contains only truncated functions with operators all at the same time. Then by Lemma 6 these terms are identically zero.

4. *If* $\alpha_{01}=1$, then $\mu\geqslant 1$. Moreover $\delta_k\geqslant 3$ for some $k=1,..., \mu$ in (33). [Suppose not. Then $\delta_j=2$ for all $j=1, 2,..., \mu$ and as $\gamma_j+\delta_j$ ($\geqslant 4$) is even for all $j=1, 2,..., \mu$, then $\gamma_j=2$ for all j. But this implies, as $\alpha_{01}=1$, that $|A|$ and $|V|$ are odd, contradicting the gauge invariance of A and V.] We also have $r=\alpha_{01}+\gamma_1+\cdots+\gamma_\mu\geqslant 2$. Then (33) is bounded by

$$\leqslant \lambda g C_\varepsilon \int_0^t dt_1\,(1+t_1)^{-\nu/2+\varepsilon}\to 0$$

in the van Hove limit. ∎

Hence there are no contributions from first order.

6.2. Second-Order Terms

In this subsection we will prove Theorem 3.

Proof of Theorem 3. The second-order terms can be written as a sum of expressions of the form

$$(i\lambda g)^2\int_0^t dt_1\int_0^{t_1} dt_2 \sum_{x_1, x_2} S([W_{2t_2}(x_2), [W_{1t_1}(x_1), A'_t]])\,\mathfrak{F}(t, t_1, t_2) \quad (34)$$

From (29), the expression (34) is bounded by

$$\begin{aligned}
&\leqslant (\lambda g)^2\, C\int_0^t dt_1\int_0^{t_1} dt_2 \sum_{x_1, x_2}\ \sum_{u_1,..., u_\mu} |K_{t-t_1}(x_1)|^{\alpha_{01}}\\
&\quad\times |K_{t-t_2}(x_2)|^{\alpha_{02}}\, |K_{t_1-t_2}(x_2-x_1)|^{\alpha_{12}}\\
&\quad\times |K_t(u_1)|^{\beta_{10}}\, |K_{t_1}(u_1-x_1)|^{\beta_{11}}\, |K_{t_2}(u_1-x_2)|^{\beta_{12}}\cdots |K_t(u_\mu)|^{\beta_{\mu 0}}\\
&\quad\times |K_{t_1}(u_\mu-x_1)|^{\beta_{\mu 1}}\, |K_{t_2}(u_\mu-x_2)|^{\beta_{\mu 2}}
\end{aligned}\quad (35)$$

where

$$|A'| = \alpha_{01} + \alpha_{02} + \beta_{10} + \cdots + \beta_{\mu 0} \geqslant 2$$
$$|V_1| = |W_1| = \alpha_{01} + \alpha_{12} + \beta_{11} + \cdots + \beta_{\mu 1} \geqslant 2$$
$$|V_2| = |W_2| = \alpha_{02} + \alpha_{12} + \beta_{12} + \cdots + \beta_{\mu 2} \geqslant 2$$

Quasifree Second-Order Terms. Consider first the quasifree case, that is, terms with two-point functions only ($\mu = 0$). Then (35) is reduced to

$$\begin{aligned} \leqslant (\lambda g)^2 C \int_0^t dt_1 \int_0^{t_1} dt_2 \sum_{x_1} |K_{t-t_1}(x_1)|^{\alpha_{01}} \\ \times \sum_{x_2} |K_{t-t_2}(x_2)|^{\alpha_{02}} |K_{t_1-t_2}(x_2 - x_1)|^{\alpha_{12}} \end{aligned} \tag{36}$$

The sum over x_2 is bounded by

$$\|K_{t-t_2}\|_p^{\alpha_{02}} \|K_{t_1-t_2}\|_p^{\alpha_{12}}$$

where $p = \alpha_{02} + \alpha_{12} = |V_2|$. Then the sum over x_1 is bounded by $\|K_{t-t_1}\|_{\alpha_{01}}^{\alpha_{01}}$. Therefore (36) is bounded by

$$(\lambda g)^2 C \int_0^t dt_1 \int_0^{t_1} dt_2 \|K_{t-t_1}\|_{\alpha_{01}}^{\alpha_{01}} \|K_{t-t_2}\|_p^{\alpha_{02}} \|K_{t_1-t_2}\|_p^{\alpha_{12}} \tag{37}$$

We can also rewrite (36) as

$$\begin{aligned} \leqslant (\lambda g)^2 C \int_0^t dt_1 \int_0^{t_1} dt_2 \sum_{x_2} |K_{t-t_2}(x_2)|^{\alpha_{02}} \\ \times \sum_{x_1} |K_{t-t_1}(x_1)|^{\alpha_{01}} |K_{t_1-t_2}(x_2 - x_1)|^{\alpha_{12}} \end{aligned} \tag{38}$$

The sum over x_1 is bounded by

$$\|K_{t-t_1}\|_q^{\alpha_{01}} \|K_{t_1-t_2}\|_q^{\alpha_{12}}$$

where $q = \alpha_{01} + \alpha_{12} = |V_1|$. Then the sum over x_2 is bounded by $\|K_{t-t_2}\|_{\alpha_{02}}^{\alpha_{02}}$. Therefore (38) is bounded by

$$(\lambda g)^2 C \int_0^t dt_1 \int_0^{t_1} dt_2 \|K_{t-t_2}\|_{\alpha_{02}}^{\alpha_{02}} \|K_{t-t_1}\|_q^{\alpha_{01}} \|K_{t_1-t_2}\|_q^{\alpha_{12}} \tag{39}$$

We can also rewrite (36) as

$$\leqslant (\lambda g)^2\, C \int_0^t dt_1 \int_0^{t_1} dt_2 \sum_{x_2} |K_{t_1-t_2}(x_2)|^{\alpha_{12}}$$

$$\times \sum_{x_1} |K_{t-t_1}(x_1)|^{\alpha_{01}}\, |K_{t-t_2}(x_2+x_1)|^{\alpha_{02}} \tag{40}$$

where we have shifted the x_1 dependence from $K_{t_1-t_2}$ to K_{t-t_2}. The sum over x_1 is bounded by

$$\|K_{t-t_1}\|_r^{\alpha_{01}}\, \|K_{t-t_2}\|_r^{\alpha_{02}}$$

where $r = \alpha_{01} + \alpha_{02} = |A'|$. Then the sum over x_2 is bounded by $\|K_{t_1-t_2}\|_{\alpha_{12}}^{\alpha_{12}}$. Therefore (40) is bounded by

$$(\lambda g)^2\, C \int_0^t dt_1 \int_0^{t_1} dt_2\, \|K_{t_1-t_2}\|_{\alpha_{12}}^{\alpha_{12}}\, \|K_{t-t_1}\|_r^{\alpha_{01}}\, \|K_{t-t_2}\|_r^{\alpha_{02}} \tag{41}$$

By Remark 10, $\alpha_{01} \geqslant 1$ and $\alpha_{02} + \alpha_{12} \geqslant 1$. Moreover, by Lemma 8, $|V_2| \geqslant 4$, so $\alpha_{02} + \alpha_{12} = |V_2| \geqslant 4$. We consider the following cases:

1. *If* $\alpha_{01} = 1$, then $\alpha_{02} \geqslant 1$, as $\alpha_{01} + \alpha_{02} = |A'| \geqslant 2$, and $\alpha_{12} \geqslant 1$, as $\alpha_{01} + \alpha_{12} = |V_1| \geqslant 2$.

(a) *If* $\alpha_{01} = 1$ *and* $\alpha_{02} = 1$, then $\alpha_{12} \geqslant 3$, and (39) is bounded by

$$\leqslant (\lambda g)^2\, C_\varepsilon \int_0^t dt_1 \int_0^{t_1} dt_2\, (1 + t_1 - t_2)^{-\nu/2+\varepsilon}$$

which is of order τ in the van Hove limit (for $\varepsilon < \nu/2 - 1$). These terms correspond to the graph of Fig. 4.

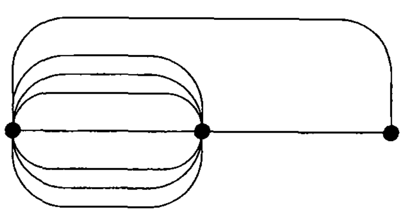

Fig. 4. Contributory terms.

(b) *If* $\alpha_{01}=1$ *and* $\alpha_{02}\geqslant 2$, then since $\alpha_{01}+\alpha_{02}=|A'|$, which is even, it follows that $\alpha_{02}\geqslant 3$. Then (39) is bounded by

$$\leqslant (\lambda g)^2 C_\varepsilon \int_0^t dt_1 \int_0^{t_2} dt_2 (1+t-t_2)^{-\nu/2+\varepsilon} \to 0$$

in the van Hove limit.

2. *If* $\alpha_{01}=2$, then if $|A'|=2$,

$$\begin{aligned}
B&=[V_1, A'] \\
&=\sum_{i=1}^{m} a(y_1-z_i+u)^* \cdots a(y_m-z_i+u)^* \\
&\quad\times a(z_1-z_i+u)\cdots a(z_{i-1}-z_i+u)\, a(v)\, a(z_{i+1}-z_i+u)\cdots a(z_m-z_i+u) \\
&\quad-\sum_{i=1}^{m} a(y_1-y_i+v)^* \cdots a(y_{i-1}-y_i+v)^*\, a(u)^* \\
&\quad\times a(y_{i+1}-y_i+v)^* \cdots a(y_m-y_i+v)^*\, a(z_1-y_i+v)\cdots a(z_m+y_i+v)
\end{aligned}$$

As $\alpha_{01}=2$, then $a(v)$ contracts with one of the $a(y_k-z_i+u)^*$'s and $a(u)^*$ with one of the $a(z_k-y_i+v)$'s. Then for each $i, j=1,..., m$ we have the factorization (up to an overall $\pm$ sign)

$$\begin{aligned}
&S(a(y_j-z_i+u)^*\, a(v))\, a(y_1-z_i+u)^* \cdots a(y_{j-1}-z_i+u)^* \\
&\qquad\times a(y_{j+1}-z_i+u)^* \cdots a(y_m-z_i+u)^* \\
&\qquad\times a(z_1-z_i+u)\cdots a(z_{i-1}-z_i+u) \\
&\qquad\times a(z_{i+1}-z_i+u)\cdots a(z_m-z_i+u)) \\
&\qquad-S(a(u)^*\, a(z_i-y_j+v))\, a(y_1-y_j+v)^* \cdots a(y_{j-1}-y_i+v)^* \\
&\qquad\times a(y_{j+1}-y_i+v)^* \cdots a(y_m-z_i+u)^* \\
&\qquad\times a(z_1-y_j+v)\cdots a(z_{i-1}-y_j+v) \\
&\qquad\times a(z_{i+1}-y_j+v)\cdots a(z_m-y_j+v)) \\
&\quad=\gamma(y_j-z_i+u-v)\{\tau_{u-z_i}-\tau_{v-y_j}\}\; C
\end{aligned}$$

where

$$\begin{aligned}
C&=a(y_1)^* \cdots a(y_{j-1})^*\, a(y_{j+1})^* \cdots a(y_m)^* \\
&\quad\times a(z_1)\cdots a(z_{i-1})\, a(z_{i+1})\cdots a(z_m)
\end{aligned}$$

But

$$S([V_2, \tau_{u-y_i} C]) = S([V_2, \tau_{v-z_j} C])$$

as V_2 is of the form $\sum_x W(x)$. Therefore such terms are identically zero. So suppose otherwise, $|A'| \geqslant 4$, that is, $\alpha_{02} \geqslant 2$. Since $\alpha_{02} + \alpha_{12} \geqslant 4$, then (37) is bounded by

$$\leqslant (\lambda g)^2 C_\varepsilon \int_0^t dt_1 \int_0^{t_1} dt_2 \, (1 + t - t_2)^{-\nu/2 + \varepsilon} \to 0$$

in the van Hove limit.

3. *If* $\alpha_{01} \geqslant 3$, then as $\alpha_{02} + \alpha_{12} \geqslant 4$, (37) is bounded by

$$\leqslant (\lambda g)^2 C_\varepsilon \int_0^t dt_1 \int_0^{t_1} dt_2 \, (1 + t - t_1)^{-\nu/2 + \varepsilon} (1 + t_1 - t_2)^{-\nu + \varepsilon} \to 0$$

in the van Hove limit (we also used $t - t_2 \geqslant t_1 - t_2$).

Non-Quasifree Second-Order Terms. The second-order terms which contain at least one higher order truncated function ($\mu \geqslant 1$) can be shown in the same manner to give no contribution in the van Hove limit. These are treated in detail in Chapter 3 of ref. 7.

Contributory Terms. In the above analysis, we have bounded the factor $|\mathfrak{F}|$ by a constant. If the remaining factors in (29) are zero in the van Hove limit, then so is the full expression. This is so in all cases except for the second-order terms of the form of Fig. 4. We will reconsider this case. These terms are bounded by

$$(\lambda g)^2 C \int_0^t dt_1 \int_0^{t_1} dt_2 \sum_{x_1, x_2} |K_{t-t_1}(x_1)| \cdot |K_{t-t_2}(x_2)| \cdot |K_{t_1-t_2}(x_2 - x_1)|^{\beta_{12}} \times |\mathfrak{F}(t, t_1, t_2)| \tag{42}$$

where $\beta_{12} \geqslant 3$. If $\mathfrak{F}$ includes a higher order function, then by Theorem 1.1

$$|\mathfrak{F}| \leqslant C_\varepsilon (1 + t_j)^{-\nu + \varepsilon} \leqslant C_\varepsilon (1 + t_2)^{-\nu + \varepsilon}$$

Therefore (42) is bounded by a constant times

$$\lambda \int_0^t dt_1 \int_0^{t_1} dt_2 \, [(1 + t_2)(1 + t_1 - t_2)]^{-\nu + \varepsilon} \to 0$$

in the van Hove limit. If, however, $\mathfrak{F}$ is composed entirely of two-point functions, then $\mathfrak{F}$ is a constant and such terms do contribute in the van Hove limit. ∎

7. BOLTZMANN EQUATION

For the remainder of the discussion we shall consider the two-body interaction (12). The contributory terms are of the form of Fig. 5 (or with the directions of all the arrows reversed) times a factor $\mathfrak{F}$ which is a product of two-point functions with creation–annihilation operators all from A.

Hugenholtz wanted to show that in the van Hove limit

$$\lim_{\lambda \to 0} S(\alpha^{\lambda g}_{\lambda^{-2}\tau} A) = \mathfrak{S}^g_\tau(A)$$

Moreover, the two-point function

$$\gamma^g_\tau(y_2 - y_1) = \mathfrak{S}^g_\tau(a(y_1)^* \, a(y_2))$$

should satisfy the nonlinear quantum Boltzmann equation[5]

$$\begin{aligned} \frac{d}{d\tau} \hat{\gamma}^g_\tau(k_0) = \pi g^2 \int dk \, dl \, dm \, [\hat{\Phi}(k - k_0) - \hat{\Phi}(k - m)]^2 \\ \times \delta(k + l - m - k_0) \, \delta(\varepsilon(k) + \varepsilon(l) - \varepsilon(m) - \varepsilon(k_0)) \\ \times \{\hat{\gamma}^g_\tau(k) \, \hat{\gamma}^g_\tau(l)[1 - \hat{\gamma}^g_\tau(m)][1 - \hat{\gamma}^g_\tau(k_0)] \\ - \hat{\gamma}^g_\tau(k_0) \, \hat{\gamma}^g_\tau(m)[1 - \hat{\gamma}^g_\tau(k)][1 - \hat{\gamma}^g_\tau(l)]\} \end{aligned} \tag{43}$$

Up to second order, (22) can be written as

$$\begin{aligned} S^{\lambda g}_{\lambda^{-2}\tau}(A) &= S(\alpha^{\lambda g}_{\lambda^{-2}\tau}(A)) \\ &= S^{(0)}_{\lambda^{-2}\tau}(A) + g\lambda S^{(1)}_{\lambda^{-2}\tau}(A) + g^2\lambda^2 S^{(2)}_{\lambda^{-2}\tau}(A) + \cdots \end{aligned}$$

Taking $\lambda \to 0$, we have the following results:

In zeroth order, this is just the free evolution:

$$\mathfrak{S}^{(0)}_\tau := \lim_{\lambda \to 0} S^{(0)}_{\lambda^{-2}\tau} = \lim_{t \to \infty} S_t = \hat{S} \tag{44}$$

by Theorem 1.

[5] Note that in (43), $[\hat{\Phi}(k - k_0) - \hat{\Phi}(k - m)]^2 \, \delta(k + l - m - k_0) \, \delta(\varepsilon(k) + \varepsilon(l) - \varepsilon(m) - \varepsilon(k_0))$ is the Born approximation to the scattering cross section.[1, 2]

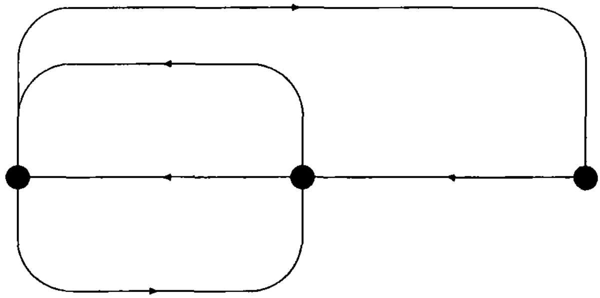

Fig. 5. Contributory terms for the two-body interaction.

In first order

$$\mathfrak{S}_\tau^{(1)}(A) := \lim_{\lambda \to 0} \lambda S^{(1)}_{\lambda^{-2}\tau}(A) = 0 \tag{45}$$

by Theorem 2.

In second order, in momentum space, the two-body interaction is [here $\hat{\Phi}(k) = (2\pi)^{-\nu} \sum_x e^{ik\cdot x}\Phi(x)$]

$$V = 1/2 \int_{-\pi}^{\pi} dk\, dl\, dm\, dn\, \hat{\Phi}(k-n)\, \delta(k+l-m-n)\, \hat{a}(k)^*\, \hat{a}(l)^*\, \hat{a}(m)\, \hat{a}(n)$$

Consider first $|A| = 2$: $A = a(f)^*\, a(g)$. The second-order term is then

$$\begin{aligned}
&[(\lambda g)^2/2]\, S^{(2)}_{\lambda^{-2}\tau}(a(f)^*\, a(g)) \\
&\quad = (\lambda g)^2/2 \int_0^{\lambda^{-2}\tau} dt_1 \int_0^{t_1} dt_2 \int dk_0\, \overline{\hat{f}(k_0)}\, \hat{g}(k_0) \\
&\qquad \times \int dk\, dl\, dm\, [\hat{\Phi}(k-k_0) - \Phi(k-m)]^2\, \delta(k+l-m-k_0) \\
&\qquad \times \{\hat{\gamma}(k)\, \hat{\gamma}(l)[1-\hat{\gamma}(m)][1-\hat{\gamma}(k_0)] \\
&\qquad - \hat{\gamma}(k_0)\, \hat{\gamma}(m)[1-\hat{\gamma}(k)][1-\hat{\gamma}(l)]\} \\
&\qquad \times \{e^{i(t_1-t_2)(\varepsilon(k)+\varepsilon(l)-\varepsilon(m)-\varepsilon(k_0))} + e^{i(t_2-t_1)(\varepsilon(k)+\varepsilon(l)-\varepsilon(m)-\varepsilon(k_0))}\}
\end{aligned} \tag{46}$$

This term corresponds to the graph in Fig. 5 (and the corresponding graph in which the directions of all arrows are reversed).

Now

$$\mathfrak{S}_\tau^{(2)}(a(f)^*\, a(g)) := \lim_{\lambda \to 0} \lambda^2 S^{(2)}_{\lambda^{-2}\tau}(a(f)^*\, a(g))$$

Hence we must evaluate the limit as $\lambda \to 0$ of (46), which is of the form

$$\lambda^2 \int_0^{\lambda^{-2}\tau} dt_1 \int_0^{t_1} dt_2\, f(t_1 - t_2) = \tau \int_0^{\lambda^{-2}\tau} ds\, f(s) - \lambda^2 \int_0^{\lambda^{-2}\tau} ds\, sf(s) \qquad (47)$$

Now [see (41) with $r = 2$, $\alpha_{12} = 3$]

$$|f(s)| \leqslant C \, \|K_s\|_3^3 \leqslant C'(1+|s|)^{-\nu/2+\delta} \leqslant C'(1+|s|)^{-3/2+\delta}$$

for $\nu \geqslant 3$. Thus $\int_0^\infty |f(s) \leqslant \infty$, and

$$\lambda^2 \int_0^{\lambda^{-2}\tau} ds\, sf(s) \leqslant c'\lambda^2 \int_0^{\lambda^{-2}\tau} ds\, (1+s)^{-1/2+\delta} \to 0$$

as $\lambda \to 0$. Hence as $\lambda \to 0$, (47) converges to $\lambda \int_0^\infty ds\, f(s)$. Consequently, using

$$\int_{-\infty}^{\infty} ds\, e^{ias} = 2\pi\delta(a)$$

we obtain

$$\begin{aligned}
&\mathfrak{S}_\tau^{(2)}(a(f)^* a(g)) \\
&\quad = \pi\tau \int dk_0\, \overline{\hat{f}(k_0)}\, \hat{g}(k_0) \int dk\, dl\, dm\, [\hat{\Phi}(k-k_0) - \Phi(k-m)]^2 \\
&\qquad \times \delta(k+l-m-k_0)\, \delta(\varepsilon(k)+\varepsilon(l)-\varepsilon(m)-\varepsilon(k_0)) \\
&\qquad \times \{\hat{\gamma}(k)\, \hat{\gamma}(l)[1-\hat{\gamma}(m)][1-\hat{\gamma}(k_0)] - \hat{\gamma}(k_0)\, \hat{\gamma}(m)[1-\hat{\gamma}(k)][1-\hat{\gamma}(l)]\} \\
&\quad = \int dk_0\, \overline{\hat{f}(k_0)}\, \hat{g}(k_0)\, \gamma_\tau^{(2)}(k_0)
\end{aligned}$$

where

$$\begin{aligned}
\dot{\gamma}_\tau^{(2)}(k_0) = \pi\tau \int dk\, dl\, dm\, &[\hat{\Phi}(k-k_0) - \hat{\Phi}(k-m)]^2 \\
&\times \delta(k+l-m-k_0)\, \delta(\varepsilon(k)+\varepsilon(l)-\varepsilon(m)-\varepsilon(k_0)) \\
&\times \{\hat{\gamma}(k)\, \hat{\gamma}(l)[1-\hat{\gamma}(m)][1-\hat{\gamma}(k_0)] \\
&- \hat{\gamma}(k_0)\, \hat{\gamma}(m)[1-\hat{\gamma}(k)][1-\hat{\gamma}(l)]\}
\end{aligned} \qquad (48)$$

Now consider a general A: $A = a(f_1)^{\#} \cdots a(f_{2m_0})^{\#}$. Since for the terms corresponding to Fig. 5, $\mathfrak{F}$ is made up only of creation–annihilation oeprators from A,

$$
\begin{aligned}
&\mathfrak{S}_\tau^{(2)}(a(f_1)^{\#} \cdots a(f_{2m_0}))\\
&\quad = \lim_{\lambda \to 0} \lambda S_{\lambda^{-2}\tau}^{(2)}(a(f_1)^{\#} \cdots a(f_{2m_0})^{\#})\\
&\quad = \sum_P \sigma(P) \sum_{l=1}^{m_0} \mathfrak{S}_\tau^{(2)}(a(f_{i_l})^* \, a(f_{j_l})) \prod_{\substack{s=1\\ s\neq l}}^{m_0} S(a(f_{i_s})^{\#} \, a(f_{j_s})^{\#})
\end{aligned} \tag{49}
$$

where the sum is taken over all partitions of the set $(1,..., 2k)$ into k pairs $(i_1, j_1),..., (i_k, j_k)$ with $i_s < j_s$, $i_1 < i_2 < \cdots < i_k$, and $\sigma(P)$ is the parity of the permutation taking $(1, 2,..., 2k)$ into $(i_1, j_1,..., i_k, j_k)$.

Assuming $\mathfrak{S}_\tau^g$ is quasifree, we expand in powers of g:

$$
\begin{aligned}
&\mathfrak{S}_\tau^g(a(f_1)^{\#} \cdots a(f_{2m_0})^{\#})\\
&\quad = \sum_P \sigma(P) \sum_{s=1}^{m_0} \mathfrak{S}_\tau^g(a(f_{i_s})^{\#} \, a(f_{j_s})^{\#})\\
&\quad = \sum_P \sigma(P) \sum_{s=1}^{m_0} \{\hat{S}(a(f_{i_s})^{\#} \, a(f_{j_s})^{\#}) + g^2 \mathfrak{S}_\tau^{(2)}(a(f_{i_s})^{\#} \, a(f_{j_s})^{\#}) + \cdots\}\\
&\quad = \sum_P \sigma(P) \sum_{s=1}^{m_0} \hat{S}(a(f_{i_s})^{\#} \, a(f_{j_s})^{\#})\\
&\qquad + g^2 \sum_{l=1}^{m_0} \sum_P \sigma(P) \left\{ \mathfrak{S}_\tau^{(2)}(a(f_{i_l})^* \, a(f_{j_l})) \prod_{\substack{s=1\\ s\neq l}}^{m_0} S(a(f_{i_s})^{\#} \, a(f_{j_s})^{\#}) \right\} + \cdots
\end{aligned} \tag{50}
$$

which agrees with (44), (45), and (49) and hence confirms that up to second order, $\mathfrak{S}_\tau^g$ is indeed quasifree. Now

$$
\begin{aligned}
\frac{d}{d\tau} \hat{\gamma}_\tau^g(k_0) &= \frac{d}{d\tau} \{\hat{\gamma}(k_0) + g^2 \hat{\gamma}_\tau^{(2)}(k_0) + \cdots\}\\
&= g^2 \frac{d}{d\tau} \hat{\gamma}_\tau^{(2)}(k_0) + \cdots
\end{aligned}
$$

$$= \pi g^2 \int dk\, dl\, dm\, [\hat{\Phi}(k-k_0) - \hat{\Phi}(k-m)]^2\, \delta(k+l-m-k_0)$$
$$\times \delta(\varepsilon(k)+\varepsilon(l)-\varepsilon(m)-\varepsilon(k_0))$$
$$\times \{\hat{\gamma}(k)\,\hat{\gamma}(l)[1-\hat{\gamma}(m)][1-\hat{\gamma}(k_0)]$$
$$-\hat{\gamma}(k_0)\,\hat{\gamma}(m)[1-\hat{\gamma}(k)][1-\hat{\gamma}(l)]\} + \cdots$$

by (48). This is just the Boltzmann equation (43) to second order in g. This proves Theorem 4.

Remark 11. A different technique is required for the analogous result in the case of the continuum $\mathbb{R}^\nu$. Instead of l^p-norms of the free evolution kernel, determinant inequalities for imaginary Gaussian integrals are used. The interaction has to be regularized so that the interaction density is bounded. A complete analysis can be found in Chapter 4 of ref. 7.

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